Squared Eigenfunctions for the Sasa-Satsuma Equation

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Squared eigenfunctions are quadratic combinations of Jost functions and adjoint Jost functions which satisfy the linearized equation of an integrable equation. They are needed for various studies related to integrable equations, such as the development of its soliton perturbation theory. In this article, squared eigenfunctions are derived for the Sasa-Satsuma equation whose spectral operator is a 3×3 system, while its linearized operator is a 2×2 system. It is shown that these squared eigenfunctions are sums of two terms, where each term is a product of a Jost function and an adjoint Jost function. The procedure of this derivation consists of two steps: first is to calculate the variations of the potentials via variations of the scattering data by the Riemann-Hilbert method. The second one is to calculate the variations of the scattering data via the variations of the potentials through elementary calculations. While this procedure has been used before on other integrable equations, it is shown here, for the first time, that for a general integrable equation, the functions appearing in these variation relations are precisely the squared eigenfunctions and adjoint squared eigenfunctions satisfying respectively the linearized equation and the adjoint linearized equation of the integrable system. This proof clarifies this procedure and provides a unified explanation for previous results of squared eigenfunctions on individual integrable equations. This procedure uses primarily the spectral operator of the Lax pair. Thus two equations in the same integrable hierarchy will share the same squared eigenfunctions (except for a time-dependent factor). In the Appendix, the squared eigenfunctions are presented for the Manakov equations whose spectral operator is closely related to that of the Sasa-Satsuma equation.

I. INTRODUCTION

Squared eigenfunctions are quadratic combinations of Jost functions and adjoint Jost functions which satisfy the linearized equation of an integrable system. This name was derived from the fact that, for many familiar integrable equations such as the KdV and NLS equations, solutions of the linearized equations are squares of Jost functions of the Lax pairs [1, 2, 3, 4, 5, 6, 7]. Squared eigenfunctions are intimately related to the integrable-equation theory. For instance, they are eigenfunctions of the recursion operator of integrable equations [8, 9, 10, 11]. They also appear as self-consistent sources of integrable equations [13, 14]. A more important application of squared eigenfunctions is in the direct soliton perturbation theory, where squared eigenfunctions and their closure relation play a fundamental role [5, 6, 12, 15].

Squared eigenfunctions have been derived for a number of integrable equations including the KdV hierarchy, the AKNS hierarchy, the derivative NLS hierarchy, the sine-Gordon equation, the massive-Thirring model, the Benjamin-Ono equation, the matrix NLS equations, the Kadomtsev-Petviashvili equation, etc. [4, 6, 9, 10, 11, 16, 17, 18]. Several techniques have been used in those derivations. One is to notice the commutability relation between the linearized operator and the recursion operator of the integrable equation [10, 11, 19]. Thus squared eigenfunctions are simply eigenfunctions of the recursion operator. The drawback of this method is that one has to first derive the recursion operator of the integrable equation and determine its eigenfunctions, which can be highly non-trivial for many equations (see [20] for instance). Another technique is to first calculate variations of the scattering data via the variations of the potentials, then use the Wronskian relations to invert this relation to get the variations of the potentials via variations of the scattering data [4]. This Wronskian-relation technique involves some ingenious steps and could run into difficulties too if the problem is sufficiently complicated [4]. The third technique is related to the second one, except that one directly calculates the variations of the potentials via variations of the scattering data by the Riemann-Hilbert method [4, 6]. This third method is general and conceptually simpler. Regarding the second and third methods however, one question which was never clarified is why the functions appearing in the variation relations are indeed squared eigenfunctions and adjoint squared eigenfunctions which satisfy respectively the linearized equation and adjoint linearized equation of an integrable system. In all known examples, this was always found to be

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true by direct verifications. But whether and why it would remain true for the general case was not known.

Recently, we were interested in the Sasa-Satsuma equation which is relevant for the propagation of ultra-short optical pulses [21, 22, 23]. This equation is integrable [23]. Two interesting features about this equation are that its solitons are embedded inside the continuous spectrum of the equation [15, 24], and their shapes can be double-humped for a wide range of soliton parameters [23]. One wonders how these double-humped solitons evolve when the equation is perturbed — a question which is interesting and significant from both physical and mathematical points of view. This question can be studied by a soliton perturbation theory. Due to the embedded nature of these solitons, external perturbations will generally excite continuous-wave radiation which is in resonance with the soliton. So a key component in the soliton perturbation theory would be to calculate this continuous-wave radiation, for which squared eigenfunctions are needed (a similar situation occurs in the soliton perturbation theory for the Hirota equation [15]). A special feature of the Sasa-Satsuma equation is that, while the linearized operator of this equation is 2×2, its spectral operator is 3×3. How to build two-component squared eigenfunctions from three-component Jost functions is an interesting and curious question. As we shall see, only certain components of the 3×3 Jost functions and their adjoints are used to construct the squared eigenfunctions and the adjoint squared eigenfunctions of the Sasa-Satsuma equation. To calculate these squared eigenfunctions for the Sasa-Satsuma equation, we have tried to directly use the first method mentioned above. The recursion operator for the Sasa-Satsuma equation has been derived recently [20]. However, that operator is quite complicated, thus its eigenfunctions are difficult to obtain. The second method mentioned above is met with difficulties too. Thus we choose the third method for the Sasa-Satsuma equation and demonstrate here certain advantages of this method.

In this paper, we further develop the third method, using the Sasa-Satsuma equation as an example. We show that the functions appearing in the expansion of the variations of the potentials are always the squared eigenfunctions which satisfy the linearized equation of an integrable system, and that the functions appearing in the formulae for the variations of scattering data are always the adjoint squared eigenfunctions which satisfy the adjoint linearized equation of an integrable system. In addition, given these two relations between the variations of the potentials and the variations of scattering data, there naturally follows the closure relation for the squared eigenfunctions and their adjoints, as well as all the inner-product relations between the squared eigenfunctions and their adjoints. Thus no longer is it necessary to grind away at calculating these inner products from the asymptotics of the Jost functions. Rather one can just read off the values of the nonzero inner products from these two variation relations. This clarifies the long-standing question regarding squared eigenfunctions in connection with the linearized integrable equation, and streamlines the third method as a general and conceptually-simple procedure for the derivation of squared eigenfunctions. We apply this method to the Sasa-Satsuma equation, and find that it readily gives the squared eigenfunctions and adjoint squared eigenfunctions. The squared eigenfunctions for the Sasa-Satsuma equation are sums of two terms, with each term being a product of a component of a Jost function and a component of an adjoint Jost function. This two-term structure of squared eigenfunctions is caused by the symmetry properties of scattering data of the Sasa-Satsuma equation. It should be noted that our derivation uses almost exclusively the spectral operator of the Lax pair, thus all integrable equations with the same spectral operator (such as the Sasa-Satsuma hierarchy) will share the same set of squared eigenfunctions as we derived here (except for a time-dependent factor which is equation-specific). An additional benefit of the method we used is that for two integrable equations with similar spectral operators, the derivation of their squared eigenfunctions will be essentially the same. Thus one can get squared eigenfunctions for one equation by minor modifications for the other equation. As an example, we demonstrate in the Appendix how squared eigenfunctions for the Manakov equations can be easily obtained by minor modifications of our calculations for the Sasa-Satsuma equation.

II.   THE RIEMANN-HILBERT PROBLEM

To start our analysis, we first formulate the Riemann-Hilbert problem for the Sasa-Satsuma equation which will be needed for later calculations. The Sasa-Satsuma equation is

$$u_t + u_{xxx} + 6|u|^2 u_x + 3u(|u|^2)_x = 0. \tag{1}$$

Its spectral (scattering) problem of the Lax pair is [23]

$$Y_x = -i\zeta \Lambda Y + QY, \tag{2}$$

where Y is a matrix function, $\Lambda = \mathrm{diag}(1, 1, -1)$,

$$Q = \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & u^* \\ -u^* & -u & 0 \end{pmatrix} \tag{3}$$

is the potential matrix, the superscript "*" represents complex conjugation, and ζ is a spectral parameter. In this paper, we always assume that the potential $u(x)$ decays to zero sufficiently fast as $x \to \pm\infty$. Notice that this matrix Q has two symmetry properties. One is that it is anti-Hermitian, i.e. $Q^\dagger = -Q$, where the superscript "$\dagger$" represents the Hermitian of a matrix. The other one is that

$$\sigma Q \sigma = Q^*, \qquad \text{where} \quad \sigma = \sigma^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

Introducing new variables

$$J = YE^{-1}, \qquad E = e^{-i\zeta \Lambda x}, \tag{5}$$

then Eq. (2) becomes

$$J_x = -i\zeta[\Lambda, J] + QJ, \tag{6}$$

where $[\Lambda, J] = \Lambda J - J\Lambda$. The matrix Jost solutions $J_\pm(x, \zeta)$ of Eq. (6) are defined by the asymptotics

$$J_\pm \to \mathbb{1}, \quad \text{as} \quad x \to \pm\infty, \tag{7}$$

where "$\mathbb{1}$" is the unit matrix. Here the subscripts in $J_\pm$ refer to which end of the x-axis the boundary conditions are set. Since $\text{tr}Q = 0$, using Abel's formula on Eq. (6), we see that $\det J_\pm = 1$ for all x. In addition, since $J_\pm E$ are both solutions of the linear equations (2), they are not independent and are linearly related by the scattering matrix $S(\zeta) = [s_{ij}(\zeta)]$:

$$J_- = J_+ E S E^{-1}, \qquad \det S = 1 \tag{8}$$

for real ζ, i.e. $\zeta \in \mathbb{R}$ (for non-real ζ, certain elements in S may not be well defined).

Due to the two symmetry properties of the potential matrix Q, the Jost solutions $J_\pm$ and the scattering matrix S satisfy the corresponding symmetry properties. One is the involutive property. Since the potential Q is anti-Hermitian, we see that $J_\pm^\dagger(\zeta^*)$ and $J_\pm^{-1}(\zeta)$ both satisfy the adjoint equation of (6). In addition, we see from Eq. (7) that $J_\pm^\dagger(\zeta^*)$ and $J_\pm^{-1}(\zeta)$ have the same large-x asymptotics, thus they are equal to each other:

$$J_\pm^\dagger(\zeta^*) = J_\pm^{-1}(\zeta). \tag{9}$$

Then in view of Eq. (8), we see that

$$S^\dagger(\zeta^*) = S^{-1}(\zeta). \tag{10}$$

To derive the other symmetry properties of the Jost solutions and the scattering matrix, we notice that due to the symmetry (4) of the potential Q, it is easy to see that $\sigma J_\pm^*(-\zeta^*)\sigma$ also satisfies Eq. (6). Then in view of the asymptotics (7), we find that Jost solutions possesses the following additional symmetry

$$J_\pm(\zeta) = \sigma J_\pm^*(-\zeta^*)\sigma. \tag{11}$$

From this symmetry and the relation (8), we see that the scattering matrix S possesses the additional symmetry

$$S(\zeta) = \sigma S^*(-\zeta^*)\sigma. \tag{12}$$

Analytical properties of the Jost solutions play a fundamental role in the Riemann-Hilbert formulation for the scattering problem (6). For convenience, we express $J_\pm$ as:

$$J_- = \Phi E^{-1}, \quad \Phi = [\phi_1, \phi_2, \phi_3], \tag{13}$$

$$J_+ = \Psi E^{-1}, \quad \Psi = [\psi_1, \psi_2, \psi_3], \tag{14}$$

where Φ and Ψ are fundamental solutions of the original spectral problem (2) and are related by the scattering matrix as

$$\Phi = \Psi S. \tag{15}$$

Then in view of the boundary conditions (7), the spectral equation (6) for $J_{\pm}$ can be rewritten into the following Volterra-type integral equations:

$$J_-(\zeta; x) = \mathbb{1} + \int_{-\infty}^{x} e^{i\zeta\Lambda(y-x)} Q(y) J_-(\zeta; y) e^{i\zeta\Lambda(x-y)} dy, \tag{16}$$

$$J_+(\zeta; x) = \mathbb{1} - \int_{x}^{\infty} e^{i\zeta\Lambda(y-x)} Q(y) J_-(\zeta; y) e^{i\zeta\Lambda(x-y)} dy. \tag{17}$$

These integral equations always have solutions when the integrals on their right hand sides converge. Due to the structure (3) of the potential Q, we easily see that Eq. (16) for the first and second columns of J_- contain only the exponential factor $e^{i\zeta(x-y)}$ which decays when ζ is in the upper half plane $\mathbb{C}_+$, and Eq. (17) for the third column of J_+ contains only the exponential factor $e^{i\zeta(y-x)}$ which also falls off for $\zeta \in \mathbb{C}_+$. Thus these three columns can be analytically extended to $\zeta \in \mathbb{C}_+$. In other words, Jost solutions

$$P_+ = [\phi_1, \phi_2, \psi_3] e^{i\zeta\Lambda x} = J_- H_1 + J_+ H_2 \tag{18}$$

are analytic in $\zeta \in \mathbb{C}_+$, where

$$H_1 = \text{diag}(1,1,0), \qquad H_2 = \text{diag}(0,0,1). \tag{19}$$

Here the subscript in P refers to which half plane the functions are analytic in. From the Volterra integral equations for P_+, we see that

$$P_+(x, \zeta) \to \mathbb{1} \quad \text{as} \quad \zeta \in \mathbb{C}_+ \to \infty. \tag{20}$$

Similarly, Jost functions $[\psi_1, \psi_2, \phi_3] e^{i\zeta\Lambda x}$ are analytic in $\zeta \in \mathbb{C}_-$, and their large-ζ asymptotics is

$$[\psi_1, \psi_2, \phi_3] e^{i\zeta\Lambda x} \to \mathbb{1} \quad \text{as} \quad \zeta \in \mathbb{C}_- \to \infty. \tag{21}$$

To obtain the analytic counterpart of P_+ in $\mathbb{C}_-$, we consider the adjoint spectral equation of (6):

$$K_x = -i\zeta[\Lambda, K] - KQ. \tag{22}$$

The inverse matrices $J_{\pm}^{-1}$ satisfy this adjoint equation. Notice that

$$J_-^{-1} = E\Phi^{-1}, \quad J_+^{-1} = E\Psi^{-1}. \tag{23}$$

Let us express Φ^{-1} and Ψ^{-1} as a collection of rows,

$$\Phi^{-1} \equiv \bar{\Phi} = \begin{bmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \\ \bar{\phi}_3 \end{bmatrix}, \qquad \Psi^{-1} \equiv \bar{\Psi} = \begin{bmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{\psi}_3 \end{bmatrix}, \tag{24}$$

where the overbar refers to the adjoint quantity, then by similar techniques as used above, we can show that the first and second rows of J_-^{-1} and the third row of J_+^{-1} are analytic in $\zeta \in \mathbb{C}_-$, i.e. adjoint Jost solutions

$$P_- = e^{-i\zeta\Lambda x} \begin{bmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \\ \bar{\psi}_3 \end{bmatrix} = H_1 J_-^{-1} + H_2 J_+^{-1} \tag{25}$$

are analytic in $\zeta \in \mathbb{C}_-$. In addition, their large-ζ asymptotics is

$$P_-(x, \zeta) \to \mathbb{1} \quad \text{as} \quad \zeta \in \mathbb{C}_- \to \infty. \tag{26}$$

Similarly, the first and second rows of J_+^{-1} and the third row of J_-^{-1}, i.e. $e^{-i\zeta x}\bar{\psi}_1$, $e^{-i\zeta x}\bar{\psi}_2$, and $e^{i\zeta x}\bar{\phi}_3$, are analytic in $\zeta \in \mathbb{C}_+$, and their large-ζ asymptotics is

$$e^{-i\zeta\Lambda x} \begin{bmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{\phi}_3 \end{bmatrix} \to \mathbb{1} \quad \text{as} \quad \zeta \in \mathbb{C}_+ \to \infty. \tag{27}$$

In view of the involution properties (9) of $J_\pm$, we see that the analytic solutions $P_\pm$ satisfy the involutive property as well:

$$P_+^\dagger(\zeta^*) = P_-(\zeta). \tag{28}$$

This property can be taken as a definition of the analytic function P_- from the known analytic function P_+.

The analytic properties of Jost functions described above have immediate implications on the analytic properties of the scattering matrix S. Let us denote

$$S^{-1}(\zeta) \equiv \bar{S}(\zeta) = [\bar{s}_{ij}(\zeta)], \tag{29}$$

then since

$$S = \Psi^{-1}\Phi = \begin{bmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{\psi}_3 \end{bmatrix} [\phi_1, \phi_2, \phi_3], \qquad \bar{S} = \Phi^{-1}\Psi = \begin{bmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \\ \bar{\phi}_3 \end{bmatrix} [\psi_1, \psi_2, \psi_3], \tag{30}$$

we see immediately that $s_{11}, s_{12}, s_{21}, s_{22}$ and $\bar{s}_{33}$ can be analytically extended to the upper half plane $\zeta \in \mathbb{C}_+$, while $\bar{s}_{11}, \bar{s}_{12}, \bar{s}_{21}, \bar{s}_{22}$ and s_{33} can be analytically extended to the lower half plane $\zeta \in \mathbb{C}_-$. In addition, their large-ζ asymptotics are

$$\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \to \mathbb{1}, \quad \bar{s}_{33} \to 1, \quad \text{as} \quad \zeta \in \mathbb{C}_+ \to \infty, \tag{31}$$

and

$$\begin{pmatrix} \bar{s}_{11} & \bar{s}_{12} \\ \bar{s}_{21} & \bar{s}_{22} \end{pmatrix} \to \mathbb{1}, \quad s_{33} \to 1, \quad \text{as} \quad \zeta \in \mathbb{C}_- \to \infty. \tag{32}$$

Hence we have constructed two matrix functions P_+ and P_- which are analytic in $\mathbb{C}_+$ and $\mathbb{C}_-$ respectively. On the real line, using Eqs. (8), (18), (25), and (29), we easily see that

$$P_-(\zeta)P_+(\zeta) = G(\zeta), \qquad \zeta \in \mathbb{R}, \tag{33}$$

where

$$G = E(H_1 + H_2 S)(H_1 + S^{-1}H_2)E^{-1} = E \begin{pmatrix} 1 & 0 & \bar{s}_{13} \\ 0 & 1 & \bar{s}_{23} \\ s_{31} & s_{32} & 1 \end{pmatrix} E^{-1}. \tag{34}$$

Equation (33) determines a matrix Riemann-Hilbert problem. The normalization condition for this Riemann-Hilbert problem can be seen from (20) and (26) as

$$P_\pm(x, \zeta) \to \mathbb{1} \quad \text{as} \quad \zeta \in \mathbb{C}_\pm \to \infty. \tag{35}$$

If this problem can be solved, then the potential Q can be reconstructed from an asymptotic expansion of its solution for large ζ. Indeed, writing P_+ as

$$P_+(x, \zeta) = \mathbb{1} + \zeta^{-1}P^{(1)}(x) + \zeta^{-2}P^{(2)}(x) + \mathcal{O}(\zeta^{-3}), \tag{36}$$

inserting it into Eq. (6) and comparing terms of the same order in ζ^{-1}, we find at $O(1)$ that

$$Q = i[\Lambda, P^{(1)}]. \tag{37}$$

Thus the potential Q can be reconstructed from $P^{(1)}$. At $O(\zeta^{-1})$, we find that

$$P_x^{(1)} = -i[\Lambda, P^{(2)}] + QP^{(1)}. \tag{38}$$

From the above two equations as well as the large-x asymptotics of $P_+(x, \zeta)$ from Eq. (7), we see that the full matrix $P^{(1)}$ is

$$P^{(1)}(x) = \frac{1}{2i} \begin{bmatrix} \int_{-\infty}^{x} |u(y)|^2 dy & \int_{-\infty}^{x} u^2(y)dy & u(x) \\ \int_{-\infty}^{x} u^{*2}(y)dy & \int_{-\infty}^{x} |u(y)|^2 dy & u^*(x) \\ u^*(x) & u(x) & 2\int_{x}^{\infty} |u(y)|^2 dy \end{bmatrix}. \tag{39}$$

This matrix, together with Eq. (36), gives the leading-order asymptotic expansions for the analytical functions P_+. Leading-order asymptotic expansions for P_- can be similarly derived.

Elements of the scattering matrix S are not all independent. Indeed, since the potential matrix Q contains only one independent function $u(x)$, matrix S should contain only one independent element as well. To get the dependence of scattering coefficients, we first notice from equation $S\bar{S} = \mathbb{1}$ that

$$s_{33}(\zeta)\bar{s}_{33}(\zeta) = 1 - s_{31}(\zeta)\bar{s}_{13}(\zeta) - s_{32}(\zeta)\bar{s}_{23}(\zeta), \quad \zeta \in \mathbb{R}. \tag{40}$$

Since s_{33} and $\bar{s}_{33}$ are analytic in $\mathbb{C}_-$ and $\mathbb{C}_+$ respectively, the above equation defines a (scalar) Riemann-Hilbert problem for $(s_{33}, \bar{s}_{33})$, under the canonical normalization conditions which can be seen from (31)-(32). Thus $(s_{33}, \bar{s}_{33})$ can be uniquely determined by the locations of their zeros in $\mathbb{C}_\pm$ together with the scattering data $(s_{31}, s_{32}, \bar{s}_{13}, \bar{s}_{23})$ on the real line $\zeta \in \mathbb{R}$ by the Plemelj formula. Similarly, from equation $S\bar{S} = \mathbb{1}$ we have

$$\begin{pmatrix} \bar{s}_{11} & \bar{s}_{12} \\ \bar{s}_{21} & \bar{s}_{22} \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \mathbb{1} - \begin{pmatrix} \bar{s}_{13} \\ \bar{s}_{23} \end{pmatrix} (s_{31}, s_{32}), \quad \zeta \in \mathbb{R}, \tag{41}$$

which defines another (matrix) Riemann-Hilbert problem for $(\bar{s}_{11}, \bar{s}_{12}, \bar{s}_{21}, \bar{s}_{22})$ and $(s_{11}, s_{12}, s_{21}, s_{22})$, under the canonical normalization conditions which can be seen from (31)-(32). Thus these analytical scattering elements can be determined by the scattering data $(s_{31}, s_{32}, \bar{s}_{13}, \bar{s}_{23})$ on the real line $\zeta \in \mathbb{R}$ as well. The scattering data $(s_{31}, s_{32}, \bar{s}_{13}, \bar{s}_{23})$ themselves are dependent on each other by the symmetry relations (10) and (12). Specifically, for $\xi \in \mathbb{R}$, $\bar{s}_{13}(\xi) = s_{31}^*(\xi), \bar{s}_{23}(\xi) = s_{32}^*(\xi)$, and $s_{32}(\xi) = s_{31}^*(-\xi)$. Thus the scattering matrix S indeed contains only a single independent element as we would expect.

In the next section, we will derive squared eigenfunctions for the Sasa-Satsuma equation. For the convenience of that derivation, we introduce the following notations:

$$\rho_1 \equiv \frac{s_{31}}{s_{33}}, \quad \rho_2 \equiv \frac{s_{32}}{s_{33}}, \quad \bar{\rho}_1 \equiv \frac{\bar{s}_{13}}{\bar{s}_{33}}, \quad \bar{\rho}_2 \equiv \frac{\bar{s}_{23}}{\bar{s}_{33}}. \tag{42}$$

Due to the symmetry conditions (10) and (12) of the scattering matrix S, we see that ρ_k and $\bar{\rho}_k$ satisfy the symmetry conditions

$$\bar{\rho}_k(\zeta) = \rho_k^*(\zeta), k = 1, 2, \quad \bar{\rho}_1(\zeta) = \rho_2(-\zeta), \quad \bar{\rho}_2(\zeta) = \rho_1(-\zeta), \quad \zeta \in \mathbb{R}. \tag{43}$$

In the next section, symmetry properties of Φ and Ψ will also be needed. Due to the symmetry conditions (9) and (11) of the Jost solutions, Φ and Ψ satisfy the symmetry conditions

$$\Phi^\dagger(\zeta^*) = \Phi^{-1}(\zeta), \quad \Psi^\dagger(\zeta^*) = \Psi^{-1}(\zeta), \tag{44}$$

and

$$\Phi_\pm(\zeta) = \sigma \Phi_\pm^*(-\zeta^*)\sigma, \quad \Psi_\pm(\zeta) = \sigma \Psi_\pm^*(-\zeta^*)\sigma. \tag{45}$$

Symmetry (44) means that

$$\phi_k^\dagger(\zeta^*) = \bar{\phi}_k(\zeta), \quad \psi_k^\dagger(\zeta^*) = \bar{\psi}_k(\zeta), \quad k = 1, 2, 3, \tag{46}$$

while symmetry (45), together with (46), means that

$$\sigma \phi_1(\zeta) = \bar{\phi}_2^T(-\zeta), \quad \sigma \phi_2(\zeta) = \bar{\phi}_1^T(-\zeta), \quad \sigma \phi_3(\zeta) = \bar{\phi}_3^T(-\zeta). \tag{47}$$

Here the superscript "T" represents the transpose of a matrix. These symmetry properties will be important for deriving the final expressions of squared eigenfunctions for the Sasa-Satsuma equation in the next section.

III. SQUARED EIGENFUNCTIONS AND THEIR CLOSURE RELATION

In this section, we calculate the variation of the potential via variations of the scattering data, then calculate variations of the scattering data via the variation of the potential. The first step will yield squared eigenfunctions, and it will be done by the Riemann-Hilbert method. The second step will yield adjoint squared eigenfunctions, and it will be done using basic relations of the spectral problem (2). For the ease of presentation, we first assume that s_{33} and $\bar{s}_{33}$ have no zeros in their respective planes of analyticity, i.e. the spectral problem (2) has no discrete eigenvalues. This facilitates the derivation of squared eigenfunctions. The results for the general case of s_{33} and $\bar{s}_{33}$ having zeros will be given in the end of this section.

A. Variation of the potential and squared eigenfunctions

In this subsection, we derive the variation of the potential via variations of the scattering data, which will readily yield the squared eigenfunctions satisfying the linearized Sasa-Satsuma equation. Our derivation will be based on the Riemann-Hilbert method.

To proceed, we define the following matrix functions

$$F_+ = P_+ \, \text{diag}(1, 1, \frac{1}{\bar{s}_{33}}), \qquad F_- = P_-^{-1} \, \text{diag}(1, 1, s_{33}). \tag{48}$$

The reason to introduce diagonal matrices with s_{33} and $\bar{s}_{33}$ in $F_\pm$ is to obtain a new Riemann-Hilbert problem (49) with a connection matrix $\tilde{G}$ which depends on ρ_k and $\bar{\rho}_k$ rather than s_{ij} and $\bar{s}_{ij}$. This way, the variation of the potential will be expressed in terms of variations in ρ_k and $\bar{\rho}_k$. When $\bar{s}_{33}$ and s_{33} have no zeros in $\mathbb{C}_+$ and $\mathbb{C}_-$ respectively, then $F_\pm$ as well as $F_\pm^{-1}$ are analytic in $\mathbb{C}_\pm$. On the real line, they are related by

$$F_+(\zeta) = F_-(\zeta)\tilde{G}(\zeta), \qquad \zeta \in \mathbb{R}, \tag{49}$$

where

$$\tilde{G} = \text{diag}(1, 1, \frac{1}{s_{33}}) \, G \, \text{diag}(1, 1, \frac{1}{\bar{s}_{33}}) = E \begin{pmatrix} 1 & 0 & \bar{\rho}_1 \\ 0 & 1 & \bar{\rho}_2 \\ \rho_1 & \rho_2 & 1 + \rho_1\bar{\rho}_1 + \rho_2\bar{\rho}_2 \end{pmatrix} E^{-1}. \tag{50}$$

Here the relation (40) has been used. Eq. (49) defines a regular Riemann-Hilbert problem (i.e. without zeros).

Next we take the variation of the Riemann-Hilbert problem (49), and get

$$\delta F_+ = \delta F_- \tilde{G} + F_- \delta\tilde{G}, \qquad \zeta \in \mathbb{R}. \tag{51}$$

Utilizing Eq. (49), we can rewrite the above equation as

$$\delta F_+ F_+^{-1} = \delta F_- F_-^{-1} + F_- \delta\tilde{G} F_+^{-1}, \qquad \zeta \in \mathbb{R}, \tag{52}$$

which defines yet another regular Riemann-Hilbert problem for $\delta F F^{-1}$. Unlike the previous Riemann-Hilbert problems (33) and (49) which were in matrix product forms, the present Riemann-Hilbert problem (52) can be explicitly solved. Using the Plemelj formula, the general solution of this Riemann-Hilbert problem is

$$\delta F F^{-1}(\zeta; x) = A_0(x) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Pi(\xi; x)}{\xi - \zeta} d\xi, \tag{53}$$

where

$$\Pi(\xi; x) \equiv F_-(\xi; x) \, \delta\tilde{G}(\xi; x) \, F_+^{-1}(\xi; x), \qquad \xi \in \mathbb{R}. \tag{54}$$

Now we consider the large-ζ asymptotics of this solution. Notice from expansions (36), (31) and relation (48) that as $\zeta \to \infty$,

$$F_+(\zeta; x) \to \begin{bmatrix} 1 + O(\frac{1}{\zeta}) & O(\frac{1}{\zeta}) & \frac{u(x)}{2i\zeta} \\ O(\frac{1}{\zeta}) & 1 + O(\frac{1}{\zeta}) & \frac{u^*(x)}{2i\zeta} \\ \frac{u^*(x)}{2i\zeta} & \frac{u(x)}{2i\zeta} & 1 + O(\frac{1}{\zeta}) \end{bmatrix}, \qquad \delta F_+(\zeta; x) \to \begin{bmatrix} O(\frac{1}{\zeta}) & O(\frac{1}{\zeta}) & \frac{\delta u(x)}{2i\zeta} \\ O(\frac{1}{\zeta}) & O(\frac{1}{\zeta}) & \frac{\delta u^*(x)}{2i\zeta} \\ \frac{\delta u^*(x)}{2i\zeta} & \frac{\delta u(x)}{2i\zeta} & O(\frac{1}{\zeta}) \end{bmatrix}. \tag{55}$$

In addition, it is easy to see that

$$\int_{-\infty}^{\infty} \frac{\Pi(\xi; x)}{\xi - \zeta} d\xi \longrightarrow -\frac{1}{\zeta} \int_{-\infty}^{\infty} \Pi(\xi; x) d\xi, \qquad \zeta \to \infty. \tag{56}$$

When these large-ζ expansions are substituted into Eq. (53), at $O(1)$, we find that $A_0(x) = 0$. At $O(\zeta^{-1})$, we get

$$\delta u(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \Pi_{13}(\xi; x) d\xi, \qquad \delta u^*(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \Pi_{31}(\xi; x) d\xi. \tag{57}$$

Now we calculate the elements Π_{13} and Π_{31}. From Eq. (50), we see that

$$\delta\tilde{G} = E \begin{pmatrix} 0 & 0 & \delta\bar{\rho}_1 \\ 0 & 0 & \delta\bar{\rho}_2 \\ \delta\rho_1 & \delta\rho_2 & \rho_1\delta\bar{\rho}_1 + \bar{\rho}_1\delta\rho_1 + \rho_2\delta\bar{\rho}_2 + \bar{\rho}_2\delta\rho_2 \end{pmatrix} E^{-1}. \tag{58}$$

Then in view of the $F_\pm$ definitions (48) and $P_\pm$ expressions (18), (25), we find from (54) that

$$\Pi(\zeta; x) = J_- E (H_1 + H_2 S)^{-1} \mathrm{diag}(1, 1, s_{33}) E^{-1} \, \delta\tilde{G} \, E \, \mathrm{diag}(1, 1, \bar{s}_{33})(H_1 + S^{-1}H_2)^{-1} E^{-1} J_-^{-1}, \tag{59}$$

which simplifies to

$$\Pi(\zeta; x) = \Phi \begin{pmatrix} 0 & 0 & \delta\bar{\rho}_1 \\ 0 & 0 & \delta\bar{\rho}_2 \\ \delta\rho_1 & \delta\rho_2 & 0 \end{pmatrix} \bar{\Phi}. \tag{60}$$

This relation, together with Eq. (57), shows that the variation of the potential δu can be expanded into quadratic combinations between Jost solutions Φ and adjoint Jost solutions $\bar{\Phi}$. This relation is generic in integrable systems.

Now we calculate explicit expressions for δu. Inserting (60) into (57) and recalling our notations (24), we readily find that

$$\delta u = -\frac{1}{\pi} \int_{-\infty}^{\infty} \left(\phi_{31}\bar{\phi}_{13}\delta\rho_1 + \phi_{31}\bar{\phi}_{23}\delta\rho_2 + \phi_{11}\bar{\phi}_{33}\delta\bar{\rho}_1 + \phi_{21}\bar{\phi}_{33}\delta\bar{\rho}_2 \right) d\xi. \tag{61}$$

Here the notations are

$$\phi_k = \begin{bmatrix} \phi_{k1} \\ \phi_{k2} \\ \phi_{k3} \end{bmatrix}, \quad \psi_k = \begin{bmatrix} \psi_{k1} \\ \psi_{k2} \\ \psi_{k3} \end{bmatrix}, \quad \bar{\phi}_k = \left[\bar{\phi}_{k1}, \bar{\phi}_{k2}, \bar{\phi}_{k3} \right], \quad \bar{\psi}_k = \left[\bar{\psi}_{k1}, \bar{\psi}_{k2}, \bar{\psi}_{k3} \right], \quad k = 1, 2, 3. \tag{62}$$

Utilizing the symmetry relations (43) and (47), the above δu formula reduces to

$$\delta u = -\frac{1}{\pi} \int_{-\infty}^{\infty} \left[\left(\phi_{31}\bar{\phi}_{13} + \phi_{33}\bar{\phi}_{12} \right) \delta\rho_1 + \left(\phi_{11}\bar{\phi}_{33} + \phi_{13}\bar{\phi}_{32} \right) \delta\bar{\rho}_1 \right] d\xi. \tag{63}$$

This is an important step in our derivation, where symmetry conditions play an important role. Regarding δu^*, its formula can be obtained by taking the complex conjugate of the above equation and simplified by using the symmetry relations (46). Defining functions

$$Z_1 = \begin{bmatrix} \phi_{31}\bar{\phi}_{13} + \phi_{33}\bar{\phi}_{12} \\ \phi_{32}\bar{\phi}_{13} + \phi_{33}\bar{\phi}_{11} \end{bmatrix}, \quad Z_2 = \begin{bmatrix} \phi_{11}\bar{\phi}_{33} + \phi_{13}\bar{\phi}_{32} \\ \phi_{12}\bar{\phi}_{33} + \phi_{13}\bar{\phi}_{31} \end{bmatrix}, \tag{64}$$

then the final expression for the variation of the potential $(\delta u, \delta u^*)^T$ is

$$\begin{bmatrix} \delta u(x) \\ \delta u^*(x) \end{bmatrix} = -\frac{1}{\pi} \int_{-\infty}^{\infty} [Z_1(\xi; x)\delta\rho_1(\xi) + Z_2(\xi; x)\delta\bar{\rho}_1(\xi)] \, d\xi. \tag{65}$$

Notice here that due to the symmetry relations (46), $Z_2(\zeta)$ is equal to $Z_1^*(\zeta^*)$ with its two components swapped. Also notice that Z_1 and Z_2 are the sum of two terms, where each term is a product of a component of a Jost function and a component of an adjoint Jost function. The two-term feature of these functions is caused by the symmetry properties of the scattering data and Jost functions, which enable us to combine terms together in the general expansion (61). In the massive Thirring model, the counterparts of functions $Z_{1,2}$ are also sums of two terms [4]. The two-term structure there is not due to symmetry properties, but rather due to the asymptotics of its Jost functions in the spectral plane. The feature of each term in Z_1 and Z_2 being a product between a Jost function and an adjoint Jost function, on the other hand, is a generic feature in integrable systems. In previous studies on many integrable equations (such as the KdV, NLS, derivative NLS, and massive Thirring equations), it was found that these functions were often "squares" or products of Jost functions themselves (thus the name "squared eigenfunctions") [1, 3, 4, 16]. This was so simply because the spectral operators in those systems were 2×2, for which the adjoint Jost functions (rows of the inverse of the Jost-function matrix) are directly proportional to Jost functions themselves. That is not generic however, and does not hold for the Sasa-Satsuma equation, or in general for integrable equations whose spectral operator is 3×3 or higher.

The derivation of (65) for the expansion of $(\delta u, \delta u^*)^T$ is an important result of this subsection. It readily gives the variations in the Sasa-Satsuma fields in terms of variations in the initial data. To show this, we first restore the time dependence in equation (65). An important fact we need to notice here is that the Jost solutions Φ as defined in (13) do not satisfy the time evolution equation of the Lax pair of the Sasa-Satsuma equation. Indeed, this time evolution equation of the Lax pair is [23]

$$Y_t = -4i\zeta^3 \Lambda Y + V(\zeta, u)Y, \tag{66}$$

where the matrix V goes to zero as x approaches infinity. Obviously the large-x asymptotics of Φ [see (7) and (13)] can not satisfy the above equation as $|x| \to \infty$ where V vanishes. But this problem can be easily fixed. Defining the "time-dependent" Jost functions

$$\Phi^{(t)} = \Phi e^{-4i\zeta^3 \Lambda t}, \tag{67}$$

then these functions satisfy *both* parts of the Lax pair, (2) and (66). The reason they now satisfy the time evolution equation (66) is due to their satisfying the asymptotic time evolution equation (66) as $|x| \to \infty$ as well as the compatibility relation of the Lax pair. Similarly we define the "time-dependent" adjoint Jost functions as

$$\bar{\Phi}^{(t)} = e^{4i\zeta^3 \Lambda t}\bar{\Phi}, \tag{68}$$

which satisfy both adjoint equations of the Lax pair. Now we use these new Jost functions and adjoint Jost functions to replace those in the definitions (64) and get "time-dependent" (Z_1, Z_2) functions

$$Z_1^{(t)} = \begin{bmatrix} \phi_{31}^{(t)}\bar{\phi}_{13}^{(t)} + \phi_{33}^{(t)}\bar{\phi}_{12}^{(t)} \\ \phi_{32}^{(t)}\bar{\phi}_{13}^{(t)} + \phi_{33}^{(t)}\bar{\phi}_{11}^{(t)} \end{bmatrix}, \quad Z_2^{(t)} = \begin{bmatrix} \phi_{11}^{(t)}\bar{\phi}_{33}^{(t)} + \phi_{13}^{(t)}\bar{\phi}_{32}^{(t)} \\ \phi_{12}^{(t)}\bar{\phi}_{33}^{(t)} + \phi_{13}^{(t)}\bar{\phi}_{31}^{(t)} \end{bmatrix}. \tag{69}$$

In view of the relations (67) and (68), we see that these "time-dependent" (Z_1, Z_2) functions are related to the original ones as

$$Z_1^{(t)} = Z_1 e^{8i\zeta^3 t}, \quad Z_2^{(t)} = Z_2 e^{-8i\zeta^3 t}. \tag{70}$$

Another fact we need to notice in conjunction with the time-restored equation of (65) is that for the Sasa-Satsuma equation, the time evolution of the scattering matrix S is given by [23]

$$S_t = -4i\xi^3[\Lambda, S]. \tag{71}$$

Thus

$$s_{33}'(t) = 0, \quad s_{31}'(t) = 8i\xi^3 s_{31}(t), \tag{72}$$

where the prime indicates differentiation with respect to t. Then recalling the definition (42) of $(\rho_1, \bar{\rho}_1)$ as well as the symmetry property (10), we obtain the time evolution of the varied scattering data as

$$\delta\rho_1(\xi, t) = \delta\rho_1(\xi, 0)e^{8i\xi^3 t}, \qquad \delta\bar{\rho}_1(\xi, t) = \delta\bar{\rho}_1(\xi, 0)e^{-8i\xi^3 t}. \tag{73}$$

Now we insert the above time-dependence relations (70) and (73) into the time-restored expansion (65), and get a new expansion

$$\begin{bmatrix} \delta u(x, t) \\ \delta u^*(x, t) \end{bmatrix} = -\frac{1}{\pi} \int_{-\infty}^{\infty} \left[Z_1^{(t)}(\xi; x, t)\delta\rho_1(\xi, 0) + Z_2^{(t)}(\xi; x, t)\delta\bar{\rho}_1(\xi, 0) \right] d\xi. \tag{74}$$

The advantages of this expansion are that the Jost functions $\Phi^{(t)}$ in $[Z_1^{(t)}, Z_2^{(t)}]$ satisfy both equations of the Lax pair, and the expansion coefficients $[\delta\rho_1, \delta\bar{\rho}_1]$ are time-independent.

Since δu and δu^* must satisfy the homogeneous linearized Sasa-Satsuma equation, it follows that the functions $[Z_1^{(t)}, Z_2^{(t)}]$ appearing in the expansion (74) must also and precisely satisfy the same linearized Sasa-Satsuma equation. A similar fact for several other integrable equations has been noted before by direct verification (see [6] for instance). Here we will give a simple proof of this fact which holds for any integrable equation where an expansion such as (74) exists. Suppose the linearized operator of the Sasa-Satsuma equation for $[u(x, t), u^*(x, t)]^T$ is $\mathcal{L}$. Since the potential

$u(x,t)$ satisfies the Sasa-Satsuma equation, and the variation of the potential $[\delta u(x,t), \delta u^*(x,t)]^T$ is infinitestimal, it follows that

$$\mathcal{L}\begin{bmatrix} \delta u(x,t) \\ \delta u^*(x,t) \end{bmatrix} = 0. \tag{75}$$

Inserting Eq. (74) into the above equation, we get

$$\int_{-\infty}^{\infty} \left[\mathcal{L}Z_1^{(t)}(\xi;x,t)\delta\rho_1(\xi,0) + \mathcal{L}Z_2^{(t)}(\xi;x,t)\delta\bar{\rho}_1(\xi,0) \right] d\xi = 0. \tag{76}$$

Since the initial variations of the scattering data $(\delta\rho_1, \delta\bar{\rho}_1)(\xi,0)$ are arbitrary and linearly independent, it follows that

$$\mathcal{L}Z_1^{(t)}(\xi;x,t) = \mathcal{L}Z_2^{(t)}(\xi;x,t) = 0 \tag{77}$$

for any $\xi \in \mathbb{R}$. Thus $Z_1^{(t)}$ and $Z_2^{(t)}$ satisfy the linearized Sasa-Satsuma equation and are therefore the squared eigenfunctions of this equation.

Now in the derivation of (74), from (52) forward, no constraints have been put on the variations of u and u^*, except for the implied assumption that the scattering data for u and u^* and its variations will exist. Whence we expect that $[\delta u, \delta u^*]^T$ in (74) can be considered to be arbitrary. If so, then due to the equality which we see in (74), we already can expect $\{Z_1^{(t)}(\xi;x,t), Z_2^{(t)}(\xi;x,t), \xi \in \mathbb{R}\}$ to form a complete set in an appropriate functional space (such as L_1).

We remark that the squared eigenfunctions given here are all linearly independent. That can be easily verified upon recognizing that from (2), one may readily construct an integro-differential eigenvalue problem, for which these squared eigenfunctions are the true eigenfunctions. Then in the usual manner it follows that these squared eigenfunctions are linearly independent. Similar for the adjoint squared eigenfunctions. The linear independence of these squared eigenfunctions can also be easily established after we have obtained the dual relations of (74) which give variations of scattering data due to arbitrary variations of the potentials (see next section).

The squared eigenfunctions $[Z_1^{(t)}, Z_2^{(t)}]$ have nice analytic properties. Indeed, recalling the analytic properties of Jost functions discussed in Sec. 2, we see that $Z_1^{(t)}(\zeta)e^{-2i\zeta x}$ is analytic in $\mathbb{C}_-$, while $Z_2^{(t)}(\zeta)e^{2i\zeta x}$ is analytic in $\mathbb{C}_+$. These analytic properties will be essential when we extend our results to the general case where s_{33} and $\bar{s}_{33}$ have zeros (see end of this section).

It should be noted that in the expressions (69) for $[Z_1^{(t)}, Z_2^{(t)}]$, if $\phi_3^{(t)}$ is replaced by other Jost functions (such as $\phi_1^{(t)}$), or if $\bar{\phi}_1^{(t)}$ is replaced by other adjoint Jost functions (such as $\bar{\phi}_2^{(t)}$), the resulting functions would still satisfy the linearized Sasa-Satsuma equation and are thus also squared eigenfunctions of this equation. These facts can be verified directly by inserting such functions into the linearized Sasa-Satsuma equation and noticing that $\Phi^{(t)}$ and $\bar{\Phi}^{(t)}$ satisfy the Lax pair (2), (66) and the adjoint Lax pair respectively. Similar results also hold for other integrable equations. However, these other squared eigenfunctions in general do not have nice analytic properties in $\mathbb{C}_\pm$ nor are they linearly independent of the set (69).

B. Variations of the scattering data and adjoint squared eigenfunctions

In this subsection, we calculate variations of the scattering data which occur due to arbitrary variations in the potentials. These formulae, together with Eq. (65) or (74), will give the "adjoint" form of the squared eigenfunctions as well as their closure relation.

We start with the spectral equation (2) for Jost functions Φ, or specifically,

$$\Phi_x = -i\zeta\Lambda\Phi + Q\Phi. \tag{78}$$

Taking the variation to this equation, we get

$$\delta\Phi_x = -i\zeta\Lambda\delta\Phi + Q\delta\Phi + \delta Q\Phi. \tag{79}$$

Recalling that $\Phi \to e^{-i\zeta\Lambda x}$ as $x \to -\infty$ [see Eqs. (7) and (13)], thus $\delta\Phi \to 0$ as $x \to -\infty$. As a result, the solution of the inhomogeneous equation (79) can be found by the method of variation of parameters as

$$\delta\Phi(\zeta;x) = \Phi(\zeta;x)\int_{-\infty}^{x} \Phi^{-1}(\zeta;y)\delta Q(y)\Phi(\zeta;y)dy. \tag{80}$$

Here

$$\delta Q = \begin{pmatrix} 0 & 0 & \delta u \\ 0 & 0 & \delta u^* \\ -\delta u^* & -\delta u & 0 \end{pmatrix} \tag{81}$$

is the variation of the potential. Now we take the limit of $x \to \infty$ in the above equation. Recalling $\Phi = \Psi S$ and the asymptotics of $\Psi \to E$ as $x \to \infty$, we see that $\Phi \to ES$, $\delta\Phi \to E\delta S$ as $x \to \infty$. Thus in this limit, the above equation becomes

$$\delta S(\xi) = S(\xi) \int_{-\infty}^{\infty} \Phi^{-1}(\xi; x)\delta Q(x)\Phi(\xi; x)dx, \qquad \xi \in \mathbb{R}. \tag{82}$$

Noticing that $S\Phi^{-1} = \Psi^{-1} \equiv \bar{\Psi}$, the above equation can be rewritten as

$$\delta S(\xi) = \int_{-\infty}^{\infty} \bar{\Psi}(\xi; x)\delta Q(x)\Phi(\xi; x)dx, \qquad \xi \in \mathbb{R}. \tag{83}$$

This formula gives variations of scattering coefficients δs_{ij} via the variation of the potential δQ. From it, the variation of the scattering data $\delta\rho_1$ can be readily found. This $\delta\rho_1$ formula contains both Jost functions Φ and adjoint Jost functions $\bar{\Psi}$. When we further express Φ in terms of Ψ through the relation (15) (so that the boundary conditions of the Jost functions involved are all set at $x = +\infty$), we obtain the expression for $\delta\rho_1$ as

$$\delta\rho_1 = \frac{1}{s_{33}^2} \int_{-\infty}^{\infty} \left[\delta u \left(\bar{\psi}_{31}\varphi_3 - \bar{\psi}_{33}\varphi_2 \right) + \delta u^* \left(\bar{\psi}_{32}\varphi_3 - \bar{\psi}_{33}\varphi_1 \right) \right] dx, \tag{84}$$

where the column vector φ is defined as

$$\varphi = [\varphi_1, \varphi_2, \varphi_3]^T \equiv \bar{s}_{22}\psi_1 - \bar{s}_{21}\psi_2. \tag{85}$$

The variation $\delta\bar{\rho}_1$ can be derived by taking the complex conjugate of the above equation and utilizing the symmetry relations (43) and (46). Defining functions

$$\bar{\varphi} = [\bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3] \equiv s_{22}\bar{\psi}_1 - s_{12}\bar{\psi}_2, \tag{86}$$

$$\Omega_1 = \begin{bmatrix} \bar{\psi}_{31}\varphi_3 - \bar{\psi}_{33}\varphi_2 \\ \bar{\psi}_{32}\varphi_3 - \bar{\psi}_{33}\varphi_1 \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} \psi_{32}\bar{\varphi}_3 - \psi_{33}\bar{\varphi}_1 \\ \psi_{31}\bar{\varphi}_3 - \psi_{33}\bar{\varphi}_2 \end{bmatrix}, \tag{87}$$

and inner products

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f^T(x)g(x)dx, \tag{88}$$

then the final expressions for variations of the scattering data $\delta\rho_1$ and $\delta\bar{\rho}_1$ are

$$\delta\rho_1(\xi) = \frac{1}{s_{33}^2(\xi)} \left\langle \Omega_1(\xi; x), \begin{bmatrix} \delta u(x) \\ \delta u^*(x) \end{bmatrix} \right\rangle, \tag{89}$$

$$\delta\bar{\rho}_1(\xi) = \frac{1}{\bar{s}_{33}^2(\xi)} \left\langle \Omega_2(\xi; x), \begin{bmatrix} \delta u(x) \\ \delta u^*(x) \end{bmatrix} \right\rangle, \tag{90}$$

where $\xi \in \mathbb{R}$.

The above two equations are associated with the expansion (65) of the previous section, where time is frozen in both cases. A small problem with these equations is that the Jost functions and adjoint Jost functions appearing in the definitions (87) of $[\Omega_1, \Omega_2]$ do not satisfy the time evolution equation (66) of the Lax pair (see the previous section). To fix this problem, we repeat the practice of the previous section and introduce "time-dependent" versions of the functions $[\Omega_1, \Omega_2]$ as

$$\Omega_1^{(t)} = \begin{bmatrix} \bar{\psi}_{31}^{(t)}\varphi_3^{(t)} - \bar{\psi}_{33}^{(t)}\varphi_2^{(t)} \\ \bar{\psi}_{32}^{(t)}\varphi_3^{(t)} - \bar{\psi}_{33}^{(t)}\varphi_1^{(t)} \end{bmatrix}, \quad \Omega_2^{(t)} = \begin{bmatrix} \psi_{32}^{(t)}\bar{\varphi}_3^{(t)} - \psi_{33}^{(t)}\bar{\varphi}_1^{(t)} \\ \psi_{31}^{(t)}\bar{\varphi}_3^{(t)} - \psi_{33}^{(t)}\bar{\varphi}_2^{(t)} \end{bmatrix}, \tag{91}$$

where $\varphi^{(t)}(\xi, x, t)$ and $\bar{\varphi}^{(t)}(\xi, x, t)$ are defined as

$$\varphi^{(t)} \equiv \bar{s}_{22}(\xi)\psi_1^{(t)} - \bar{s}_{21}(\xi)\psi_2^{(t)}, \quad \bar{\varphi}^{(t)} \equiv s_{22}(\xi)\bar{\psi}_1^{(t)} - s_{12}(\xi)\bar{\psi}_2^{(t)}, \tag{92}$$

$\Phi^{(t)}$ and $\bar{\Phi}^{(t)}$ have been defined in (67)-(68), and $\Psi^{(t)}$ and $\bar{\Psi}^{(t)}$ are defined as

$$\Psi^{(t)} = \Psi e^{-4i\zeta^3 \Lambda t}, \quad \bar{\Psi}^{(t)} = e^{4i\zeta^3 \Lambda t} \bar{\Psi}. \tag{93}$$

Notice from (71) and the symmetry condition (10) that $(s_{12}, s_{22}, \bar{s}_{21}, \bar{s}_{22})$ are time-independent. Similar to what we have done in the previous section, we can show that $\varphi^{(t)}$ and $\Psi^{(t)}$ in the definition (91) are Jost functions satisfying both the Lax pair (2) and (66), while functions $\bar{\varphi}^{(t)}$ and $\bar{\Psi}^{(t)}$ satisfy both the adjoint Lax pair. In addition, $[\Omega_1^{(t)}, \Omega_2^{(t)}]$ are related to $[\Omega_1, \Omega_2]$ as

$$\Omega_1^{(t)} = \Omega_1 e^{-8i\zeta^3 t}, \quad \Omega_2^{(t)} = \Omega_2 e^{8i\zeta^3 t}. \tag{94}$$

Inserting this relation and (73) into the time-restored equations (89) and (90), we get the new relations

$$\delta\rho_1(\xi, 0) = \frac{1}{s_{33}^2(\xi)} \left\langle \Omega_1^{(t)}(\xi; x, t), \begin{bmatrix} \delta u(x, t) \\ \delta u^*(x, t) \end{bmatrix} \right\rangle, \tag{95}$$

$$\delta\bar{\rho}_1(\xi, 0) = \frac{1}{\bar{s}_{33}^2(\xi)} \left\langle \Omega_2^{(t)}(\xi; x, t), \begin{bmatrix} \delta u(x, t) \\ \delta u^*(x, t) \end{bmatrix} \right\rangle. \tag{96}$$

These two new relations are associated with the new expansion (74) in the previous section.

The functions $(\Omega_1^{(t)}, \Omega_2^{(t)})$ appearing in the above variations-of-scattering-data formulae (95)-(96) are precisely adjoint squared eigenfunctions satisfying the adjoint linearized Sasa-Satsuma equation. Analogous facts for several other integrable equations have been noted before by direct verification (see [6]). Below we will present a general proof of this fact which will hold for those integrable equations where we have relations similar to (74), (95) and (96). Along the way, we will also obtain the closure relation, the orthogonality relations and inner products between the squared eigenfunctions and the adjoint squared eigenfunctions.

First, we insert Eqs. (95)-(96) into the expansion (74). Exchanging the order of integration, we get

$$\begin{bmatrix} \delta u(x, t) \\ \delta u^*(x, t) \end{bmatrix} = -\frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\frac{1}{s_{33}^2(\xi)} Z_1^{(t)}(\xi; x, t)\Omega_1^{(t)T}(\xi; x', t) + \frac{1}{\bar{s}_{33}^2(\xi)} Z_2^{(t)}(\xi; x, t)\Omega_2^{(t)T}(\xi; x', t) \right] d\xi \begin{bmatrix} \delta u(x', t) \\ \delta u^*(x', t) \end{bmatrix} dx'. \tag{97}$$

Since we are considering that $(\delta u, \delta u^*)$ are arbitrary localized functions (in an appropriate functional space such as L_1), in order for the above equation to hold, we must have

$$-\frac{1}{\pi} \int_{-\infty}^{\infty} \left[\frac{1}{s_{33}^2(\xi)} Z_1^{(t)}(\xi; x, t)\Omega_1^{(t)T}(\xi; x', t) + \frac{1}{\bar{s}_{33}^2(\xi)} Z_2^{(t)}(\xi; x, t)\Omega_2^{(t)T}(\xi; x', t) \right] d\xi = \delta(x - x')\mathbb{1}, \tag{98}$$

which is the closure relation (discrete-spectrum contributions are absent here due to our assumption of s_{33}, $\bar{s}_{33}$ having no zeros). Here $\delta(x)$ is the dirac delta function. This closure relation has the usually symmetry in that not only do the squared eigenfunctions $\{Z_1^{(t)}(\xi; x, t), Z_2^{(t)}(\xi; x, t), \xi \in \mathbb{R}\}$ form a complete set, but also the adjoint functions $\{\Omega_1^{(t)}(\xi; x, t), \Omega_2^{(t)}(\xi; x, t), \xi \in \mathbb{R}\}$ form a complete set as well.

To obtain the inner products and orthogonality relations between these functions, we reverse the above and insert the expansion (74) into Eqs. (95)-(96). Exchanging the order of integration, we get

$$\begin{bmatrix} \delta\rho_1(\xi, 0) \\ \delta\bar{\rho}_1(\xi, 0) \end{bmatrix} = \int_{-\infty}^{\infty} M(\xi, \xi', t) \begin{bmatrix} \delta\rho_1(\xi', 0) \\ \delta\bar{\rho}_1(\xi', 0) \end{bmatrix} d\xi', \tag{99}$$

where

$$M(\xi, \xi', t) = \begin{pmatrix} -\frac{1}{\pi s_{33}^2(\xi)} \left\langle \Omega_1^{(t)}(\xi; x, t), Z_1^{(t)}(\xi'; x, t) \right\rangle, & -\frac{1}{\pi s_{33}^2(\xi)} \left\langle \Omega_1^{(t)}(\xi; x, t), Z_2^{(t)}(\xi'; x, t) \right\rangle \\ -\frac{1}{\pi \bar{s}_{33}^2(\xi)} \left\langle \Omega_2^{(t)}(\xi; x, t), Z_1^{(t)}(\xi'; x, t) \right\rangle, & -\frac{1}{\pi \bar{s}_{33}^2(\xi)} \left\langle \Omega_2^{(t)}(\xi; x, t), Z_2^{(t)}(\xi'; x, t) \right\rangle \end{pmatrix}. \tag{100}$$

Since the relation (99) is taken to be valid for arbitrary functions of $\delta\rho_1(\xi,0)$ and $\delta\bar{\rho}_1(\xi,0)$, thus

$$M(\xi,\xi',t) = \mathbb{1}\delta(\xi - \xi'). \tag{101}$$

Consequently we get the inner products and orthogonality relations between squared eigenfunctions $[Z_1^{(t)}, Z_2^{(t)}]$ and functions $[\Omega_1^{(t)}, \Omega_2^{(t)}]$ as

$$\left\langle \Omega_1^{(t)}(\xi;x,t), Z_1^{(t)}(\xi';x,t) \right\rangle = -\pi s_{33}^2(\xi)\delta(\xi - \xi'), \tag{102}$$

$$\left\langle \Omega_2^{(t)}(\xi;x,t), Z_2^{(t)}(\xi';x,t) \right\rangle = -\pi \bar{s}_{33}^2(\xi)\delta(\xi - \xi'), \tag{103}$$

$$\left\langle \Omega_1^{(t)}(\xi;x,t), Z_2^{(t)}(\xi';x,t) \right\rangle = \left\langle \Omega_2^{(t)}(\xi;x,t), Z_1^{(t)}(\xi';x,t) \right\rangle = 0 \tag{104}$$

for any $\xi, \xi' \in \mathbb{R}$.

To show that (Ω_1, Ω_2) are adjoint squared eigenfunctions which satisfy the adjoint linearized equation of the Sasa-Satsuma equation, we first separate the temporal derivatives from the spatial ones in the linearized operator $\mathcal{L}$ as

$$\mathcal{L} = \mathbb{1}\partial_t + L, \tag{105}$$

where L only involves spatial derivatives. The adjoint linearized operator is then

$$\mathcal{L}^A = -\mathbb{1}\partial_t + L^A, \tag{106}$$

where L^A is the adjoint operator of L. Now we take the inner product between the equation $\mathcal{L}Z_1^{(t)}(\xi';x,t) = 0$ and function $\Omega_1^{(t)}(\xi;x,t)$. When Eq. (105) is inserted into it, we find that

$$\langle LZ_1^{(t)}(\xi'), \Omega_1^{(t)}(\xi) \rangle - \langle Z_1^{(t)}(\xi'), \partial_t\Omega_1^{(t)}(\xi) \rangle + \partial_t\langle Z_1^{(t)}(\xi'), \Omega_1^{(t)}(\xi) \rangle = 0. \tag{107}$$

Here and immediately below, the (x,t) dependence of $Z_{1,2}^{(t)}$ and $\Omega_{1,2}^{(t)}$ is suppressed for notational simplicity. Recall that s_{33} is time-independent for the Sasa-Satsuma equation, thus from the inner-product equation (102) we see that $\partial_t\langle Z_1^{(t)}(\xi'), \Omega_1^{(t)}(\xi) \rangle = 0$. Using integration by parts, the first term in the above equation can be rewritten as

$$\langle LZ_1^{(t)}(\xi'), \Omega_1^{(t)}(\xi) \rangle = \langle Z_1^{(t)}(\xi'), L^A\Omega_1^{(t)}(\xi) \rangle + W(\xi',\xi)|_{x=-L}^{x=L}, \qquad L \to \infty, \tag{108}$$

where function $W(\xi',\xi;x,t)$ contains terms which are generated during integration by parts. These terms are quadratic combinations of $Z_1^{(t)}(\xi')$, $\Omega_1^{(t)}(\xi)$ and their spatial derivatives. When $x = \pm L \to \pm\infty$, each term has the form $f(\xi')g(\xi)\exp(\pm i\xi' L \pm i\xi L \pm 4i\xi'^3 t \pm 4i\xi^3 t)$ due to the large-x asymptotics of Jost functions Φ and Ψ. Here $f(\xi')$ and $g(\xi)$ are related to scattering coefficients and are continuous functions. Then in the sense of generalized functions, the last term in Eq. (108) is zero due to the Riemann-Lebesgue lemma. Inserting the above results into Eq. (107), we find that

$$\langle Z_1^{(t)}(\xi'), \mathcal{L}^A\Omega_1^{(t)}(\xi) \rangle = 0, \qquad \xi', \xi \in \mathbb{R}, \tag{109}$$

i.e. $\mathcal{L}^A\Omega_1^{(t)}(\xi)$ is orthogonal to all $Z_1^{(t)}(\xi')$. By taking the inner product between the equation $\mathcal{L}Z_2^{(t)}(\xi') = 0$ and function $\Omega_1^{(t)}(\xi)$ and doing similar calculations, we readily find that

$$\langle Z_2^{(t)}(\xi'), \mathcal{L}^A\Omega_1^{(t)}(\xi) \rangle = 0, \qquad \xi', \xi \in \mathbb{R}. \tag{110}$$

In other words, $\mathcal{L}^A\Omega_1^{(t)}(\xi)$ is also orthogonal to all $Z_2^{(t)}(\xi')$. Since $\{Z_1^{(t)}(\xi'), Z_2^{(t)}(\xi'), \xi' \in \mathbb{R}\}$ forms a complete set, Eqs. (109)-(110) dictate that $\mathcal{L}^A\Omega_1^{(t)}$ has to be zero for any ξ, x and t, i.e.

$$\mathcal{L}^A\Omega_1^{(t)}(\xi;x,t) = 0, \qquad \xi \in \mathbb{R}. \tag{111}$$

Thus $\Omega_1^{(t)}(\xi)$ is an adjoint squared eigenfunction that satisfies the adjoint linearized Sasa-Satsuma equation for every $\xi \in \mathbb{R}$. Similarly, it can be shown that $\Omega_2^{(t)}(\xi)$ is also an adjoint squared eigenfunction for every $\xi \in \mathbb{R}$. In short, $\{\Omega_1^{(t)}(\xi), \Omega_2^{(t)}(\xi), \xi \in \mathbb{R}\}$ is a complete set of adjoint squared eigenfunctions.

Like squared eigenfunctions $(Z_1^{(t)}, Z_2^{(t)})$, these adjoint squared eigenfunctions $(\Omega_1^{(t)}, \Omega_2^{(t)})$ are also quadratic combinations of Jost functions and adjoint Jost functions. In addition, they have nice analytic properties as well. To see this latter fact, notice from the definition (85) that $\varphi e^{i\zeta x}$ is analytic in $\mathbb{C}_-$ since $\bar{s}_{21}, \bar{s}_{22}, \psi_1 e^{i\zeta x}$ and $\psi_2 e^{i\zeta x}$ are analytic in $\mathbb{C}_-$. Then recalling that $\bar{\psi}_3 e^{i\zeta x}$ is also analytic in $\mathbb{C}_-$, we see that $\Omega_1^{(t)}(\zeta)e^{2i\zeta x}$ is analytic in $\mathbb{C}_-$. Similarly, $\Omega_2^{(t)}(\zeta)e^{-2i\zeta x}$ is analytic in $\mathbb{C}_+$.

It is worthy to point out that in the above adjoint squared eigenfunctions (91), if $\varphi^{(t)}$ is replaced by the Jost function $\psi_1^{(t)}$ (or $\psi_2^{(t)}$), or if $\bar{\varphi}^{(t)}$ is replaced by the adjoint Jost function $\bar{\psi}_1^{(t)}$ (or $\bar{\psi}_2^{(t)}$), the resulting functions would still satisfy the adjoint linearized Sasa-Satsuma equation. Thus one may be tempted to simply take $\varphi^{(t)} = \psi_k^{(t)}$, $\bar{\varphi}^{(t)} = \bar{\psi}_k^{(t)}$ ($k = 1$ or 2) rather than (92) as adjoint squared eigenfunctions. The problem with these "simpler" adjoint squared eigenfunctions is that their inner products with squared eigenfunctions (69) are not what they are supposed to be [see (102)-(104)], neither does one get the closure relation (98). Thus the adjoint squared eigenfunctions (91) coming from our systematic procedure of variation calculations are the correct ones to use.

C.   Extension to the general case

In the previous subsections, the squared eigenfunctions and their closure relation were established under the assumption that s_{33} and $\bar{s}_{33}$ have no zeros, i.e. the spectral equation (2) has no discrete eigenvalues. In this subsection, we extend those results to the general case where s_{33} and $\bar{s}_{33}$ have zeros in their respective planes of analyticity $\mathbb{C}_\pm$. Due to the symmetry properties (10) and (12), we have

$$\bar{s}_{33}(\zeta) = s_{33}^*(\zeta^*), \quad s_{33}(\zeta) = s_{33}^*(-\zeta^*). \tag{112}$$

Thus if $\zeta_j \in \mathbb{C}_-$ is a zero of s_{33}, i.e. $s_{33}(\zeta_j) = 0$, then $-\zeta_j^* \in \mathbb{C}_-$ is also a zero of s_{33}, and $\{\zeta_j^*, -\zeta_j\} \in \mathbb{C}_+$ are both zeros of $\bar{s}_{33}$. This means that zeros of s_{33} and $\bar{s}_{33}$ always appear in quadruples. Suppose all the zeros of s_{33} and $\bar{s}_{33}$ are $\zeta_j \in \mathbb{C}_-$ and $\bar{\zeta}_j \in \mathbb{C}_+$, $j = 1, ..., 2N$, respectively. For simplicity, we also assume that all zeros are simple.

In this general case with zeros, squared eigenfunctions $Z_1^{(t)}, Z_2^{(t)}$ clearly still satisfy the linearized Sasa-Satsuma equation, and adjoint squared eigenfunctions $\Omega_1^{(t)}, \Omega_2^{(t)}$ still satisfy the adjoint linearized Sasa-Satsuma equation — facts which will not change when s_{33} and $\bar{s}_{33}$ have zeros. The main difference from the previous no-zero case is that, the sets of (continuous) squared eigenfunctions $\{Z_1^{(t)}(\xi; x), Z_2^{(t)}(\xi; x), \xi \in \mathbb{R}\}$ and adjoint squared eigenfunctions $\{\Omega_1^{(t)}(\xi; x), \Omega_2^{(t)}(\xi; x), \xi \in \mathbb{R}\}$ are no longer complete, i.e. the closure relation (98) does not hold any more, and contributions from the discrete spectrum must be included now. To account for discrete-spectrum contributions, one could proceed by adding variations in the discrete scattering data in the above derivations. But a much easier way is to simply pick up the pole contributions to the integrals in the closure relation (98), as we will do below. Recall from the previous subsections that $Z_1^{(t)}(\zeta; x, t)e^{-2i\zeta x}$ and $\Omega_1^{(t)}(\zeta; x, t)e^{2i\zeta x}$ are analytic in $\mathbb{C}_-$, while $Z_2^{(t)}(\zeta; x, t)e^{2i\zeta x}$ and $\Omega_2^{(t)}(\zeta; x, t)e^{-2i\zeta x}$ are analytic in $\mathbb{C}_+$. In addition, from the large-ζ asymptotics (31)-(32), (35), and symmetry relations (46), we easily find that

$$Z_1^{(t)}(\zeta; x, t) \to e^{2i\zeta x + 8i\zeta^3 t}(0, 1)^T, \qquad Z_2^{(t)}(\zeta; x, t) \to e^{-2i\zeta x - 8i\zeta^3 t}(1, 0)^T, \qquad \zeta \to \infty, \tag{113}$$

$$\Omega_1^{(t)}(\zeta; x, t) \to -e^{-2i\zeta x - 8i\zeta^3 t}(0, 1)^T, \qquad \Omega_2^{(t)}(\zeta; x, t) \to -e^{2i\zeta x + 8i\zeta^3 t}(1, 0)^T, \qquad \zeta \to \infty. \tag{114}$$

Thus,

$$\int_{C^-} \frac{1}{s_{33}^2(\zeta)} Z_1^{(t)}(\zeta; x, t)\Omega_1^{(t)T}(\zeta; x', t)d\zeta = -\int_{C^-} e^{2i\zeta(x-x')}d\zeta \, \text{diag}(0, 1), \tag{115}$$

where the integration path C^- is the lower semi-circle of infinite radius in anti-closewise direction. Since the function $e^{2i\zeta(x-x')}$ in the right hand side of the above equation is analytic, its integration path can be brought up to the real axis $\zeta \in \mathbb{R}$. In the sense of generalized functions, that integral is equal to $\pi\delta(x - x')$, thus

$$\int_{C^-} \frac{1}{s_{33}^2(\zeta)} Z_1^{(t)}(\zeta; x, t)\Omega_1^{(t)T}(\zeta; x', t)d\zeta = -\pi\delta(x - x')\text{diag}(0, 1). \tag{116}$$

Doing the same calculation for the integral of $Z_2^{(t)}\Omega_2^{(t)T}/\bar{s}_{33}^2$ along the upper semi-circle of infinite radius C^+ in closewise direction and combining it with the above equation, we get

$$-\frac{1}{\pi}\int_{C^-} \frac{1}{s_{33}^2(\zeta)} Z_1^{(t)}(\zeta; x, t)\Omega_1^{(t)T}(\zeta; x', t)d\zeta - \frac{1}{\pi}\int_{C^+} \frac{1}{\bar{s}_{33}^2(\zeta)} Z_2^{(t)}(\zeta; x, t)\Omega_2^{(t)T}(\zeta; x', t)d\zeta = \delta(x - x')\mathbb{1}. \tag{117}$$

Now we evaluate the two integrals in the above equation by bringing down the integration paths to the real axis and picking up pole contributions by the residue theorem. Recalling that zeros of s_{33} and $\bar{s}_{33}$ are simple, the poles in the above integrals are second-order. After simple calculations, we get the closure relation for the general case (with discrete-spectrum contributions) as:

$$
-\frac{1}{\pi} \int_{-\infty}^{\infty} \left[\frac{1}{s_{33}^2(\xi)} Z_1^{(t)}(\xi;x,t)\Omega_1^{(t)T}(\xi;x',t) + \frac{1}{\bar{s}_{33}^2(\xi)} Z_2^{(t)}(\xi;x,t)\Omega_2^{(t)T}(\xi;x',t) \right] d\xi
$$

$$
- \sum_{j=1}^{2N} \frac{2i}{s_{33}'^2(\bar{\zeta}_j)} \left[Z_1^{(t)}(\bar{\zeta}_j;x,t)\Theta_1^{(t)T}(\bar{\zeta}_j;x',t) + \frac{\partial Z_1^{(t)}}{\partial \zeta}(\bar{\zeta}_j;x,t)\Omega_1^{(t)T}(\bar{\zeta}_j;x',t) \right]
$$

$$
+ \sum_{j=1}^{2N} \frac{2i}{\bar{s}_{33}'^2(\zeta_j)} \left[Z_2^{(t)}(\zeta_j;x,t)\Theta_2^{(t)T}(\zeta_j;x',t) + \frac{\partial Z_2^{(t)}}{\partial \zeta}(\zeta_j;x,t)\Omega_2^{(t)T}(\zeta_j;x',t) \right] = \delta(x-x')\mathbb{1}, \qquad (118)
$$

where

$$
\Theta_1^{(t)}(\bar{\zeta}_j;x,t) = \frac{\partial \Omega_1^{(t)}}{\partial \zeta}(\bar{\zeta}_j;x,t) - \frac{s_{33}''(\bar{\zeta}_j)}{s_{33}'(\bar{\zeta}_j)} \Omega_1^{(t)}(\bar{\zeta}_j;x,t), \qquad (119)
$$

$$
\Theta_2^{(t)}(\zeta_j;x,t) = \frac{\partial \Omega_2^{(t)}}{\partial \zeta}(\zeta_j;x,t) - \frac{\bar{s}_{33}''(\zeta_j)}{\bar{s}_{33}'(\zeta_j)} \Omega_2^{(t)}(\zeta_j;x,t). \qquad (120)
$$

which have been termed "derivative states" [1, 2] due to the differentiation with respect to ζ. Clearly, $Z_1^{(t)}(\bar{\zeta}_j)$, $\partial Z_1^{(t)}/\partial \zeta \, (\bar{\zeta}_j)$, $Z_2^{(t)}(\zeta_j)$, and $\partial Z_2^{(t)}/\partial \zeta \, (\zeta_j)$ satisfy the linearized Sasa-Satsuma equation and are thus discrete squared eigenfunctions, while $\Omega_1^{(t)}(\bar{\zeta}_j)$, $\Theta_1^{(t)}(\bar{\zeta}_j)$, $\Omega_2^{(t)}(\zeta_j)$, and $\Theta_2^{(t)}(\zeta_j)$ satisfy the adjoint linearized Sasa-Satsuma equation and are thus discrete adjoint squared eigenfunctions.

In soliton perturbation theories, explicit expressions for squared eigenfunctions under soliton potentials are needed. For such potentials, $s_{31} = s_{32} = \bar{s}_{13} = \bar{s}_{23} = 0$, thus $G = \mathbb{1}$ in the Riemann-Hilbert problem (33). Solutions of this Riemann-Hilbert problem have been solved completely for any number of zeros and any orders of their algebraic and geometric multiplicities [25]. In the simplest case where all zeros of the solitons are simple, solutions $P_\pm$ can be found in Eq. (76) of [25] (setting $r^{(i} = r^{(j)} = 1$). One only needs to keep in mind here that the zeros of s_{33} and $\bar{s}_{33}$ always appear in quadruples due to the symmetries (112). Once solutions $P_\pm$ are obtained, the solitons $u(x,t)$ follow from Eq. (37). Taking the large-x asymptotics of $P_\pm$ and utilizing the symmetry conditions (10), the full scattering matrix S can be readily obtained. Then together with $P_\pm$, explicit expressions for all the Jost functions Φ, Ψ as well as squared eigenfunctions are then derived.

IV. SUMMARY AND DISCUSSION

In this paper, squared eigenfunctions were derived for the Sasa-Satsuma equation. It was shown that these squared eigenfunctions are sums of two terms, where each term is a product of a Jost function and an adjoint Jost function. The procedure of this derivation consists of two steps: one is to calculate the variations of the potentials via variations of the scattering data by the Riemann-Hilbert method. The other one is to calculate variations of the scattering data via variations of the potentials through elementary calculations. It was proved that the functions appearing in these variation relations are precisely the squared eigenfunctions and adjoint squared eigenfunctions satisfying respectively the linearized equation and the adjoint linearized equation of the Sasa-Satsuma system. More importantly, our proof is quite general and generally holds for other integrable equations. Since the spectral operator of the Sasa-Satsuma equation is 3×3, while its linearized operator is 2×2, we demonstrated how the two-component squared eigenfunctions are built from only selected components of the three-component Jost functions, and we have seen that symmetry properties of Jost functions and the scattering data play an important role here.

The derivation used in this paper for squared eigenfunctions is a universal technique. Our main contributions to this method are triple-fold. First, we showed that for a general integrable system, if one can obtain the variational relations between the potentials and the scattering data in terms of the components of the Jost functions and their adjoints, then from these variational relations one can immediately identify the squared eigenfunctions and adjoint squared eigenfunctions which satisfy respectively the linearized integrable equation and its adjoint equation. Second, we showed that from these variational relations, one can read off the values of nonzero inner products between squared eigenfunctions and adjoint squared eigenfunctions. Thus no longer is it necessary to calculate these inner

products from the Wronskian relations and the asymptotics of the Jost functions. Thirdly, we showed that from these variational relations, one can immediately obtain the closure relation of squared eigenfunctions and adjoint squared eigenfunctions. After squared eigenfunctions of an integrable equation are obtained, one then can readily derive the recursion operator for that integrable equation, since the squared eigenfunctions are eigenfunctions of the recursion operator [8, 10, 11, 19].

An important remark about this derivation of squared eigenfunctions is that it uses primarily the spectral operator of the Lax pair. Indeed, the key variation relations (65), (89) and (90) were obtained exclusively from the spectral operator (2), while the "time-dependent" versions of these relations (74), (95) and (96) were obtained by using the asymptotic form of the time-evolution equation in the Lax pair as $|x| \to \infty$. This means that all integrable equations with the same spectral operator would share the same squared eigenfunctions (except an exponential-in-time factor which is equation-dependent). For instance, a whole hierarchy of integrable equations would possess the "same" squared eigenfunctions, since the spectral operators of a hierarchy are the same. This readily reproduces earlier results in [10, 11, 19], where such results were obtained by the commutability relations between the linearized operators and the recursion operator of a hierarchy.

Since the derivation in this paper uses primarily the spectral operator of the Lax pair, if two integrable equations have similar spectral operators, derivations of their squared eigenfunctions would be similar. For instance, the spectral operator of the Manakov equations is very similar to that of the Sasa-Satsuma equation (except that the potential term Q in the Sasa-Satsuma spectral operator (2) possesses one more symmetry). Making very minor modifications to the calculations for the Sasa-Satsuma equation in the text, we can obtain squared eigenfunctions for the Manakov equations. This will be demonstrated in the Appendix.

With the squared eigenfunctions obtained for the Sasa-Satsuma equation, a perturbation theory for Sasa-Satsuma solitons can now be developed. This problem lies outside the scope of the present article, and will be left for future studies.

Acknowledgment

Appendix: Squared eigenfunctions for the Manakov equation

The Manakov equations are [26]

$$iu_t + u_{xx} + 2(|u|^2 + |v|^2)u = 0, \tag{121}$$

$$iv_t + v_{xx} + 2(|u|^2 + |v|^2)v = 0. \tag{122}$$

The spectral operator for the Manakov equations is Eq. (2) with

$$Q = \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & v \\ -u^* & -v^* & 0 \end{pmatrix}. \tag{123}$$

Below we will use the same notations on Jost functions and the scattering matrix as in the main text. Since Q above is also anti-Hermitian, Jost functions and the scattering matrix satisfy the involution properties (9) and (10) as well. The main difference between the Sasa-Satsuma equation and the Manakov equations is that, the potential Q above for the Manakov equations does not possess the additional symmetry (4), thus the Jost functions and the scattering matrix do not possess the symmetries (11) and (12). In addition, the asymptotics (55) also needs minor modification. It is noted that the spectral operator of the Manakov equations is 3×3, while the linearized operator of these equations is 4×4. So one needs to build four-component squared eigenfunctions from three-component Jost functions. This contrasts the Sasa-Satsuma equation where one builds two-component squared eigenfunctions from three-component Jost functions.

Repeating the same calculations as in the main text with the above minor modifications kept in mind, we still get Eq. (57) for the variations $(\delta u, \delta u^*)$, while the expressions for variations $(\delta v, \delta v^*)$ are Eq. (57) with Π_{13} and Π_{31}

replaced by Π_{23} and Π_{32}. The expression for matrix Π is still given by Eq. (60). Thus variations of the potentials via variations of the scattering data are found to be

$$[\delta u, \delta v, \delta u^*, \delta v^*]^T = -\frac{1}{\pi}\int_{-\infty}^{\infty}[Z_1(\xi;x)\delta\rho_1(\xi) + Z_2(\xi;x)\delta\rho_2(\xi) + Z_3(\xi;x)\delta\bar{\rho}_1(\xi) + Z_4(\xi;x)\delta\bar{\rho}_2(\xi)]\,d\xi, \qquad (124)$$

where

$$Z_1 = \begin{bmatrix} \phi_{31}\bar{\phi}_{13} \\ \phi_{32}\bar{\phi}_{13} \\ \phi_{33}\bar{\phi}_{11} \\ \phi_{33}\bar{\phi}_{12} \end{bmatrix}, \quad Z_2 = \begin{bmatrix} \phi_{31}\bar{\phi}_{23} \\ \phi_{32}\bar{\phi}_{23} \\ \phi_{33}\bar{\phi}_{21} \\ \phi_{33}\bar{\phi}_{22} \end{bmatrix}, \quad Z_3 = \begin{bmatrix} \phi_{11}\bar{\phi}_{33} \\ \phi_{12}\bar{\phi}_{33} \\ \phi_{13}\bar{\phi}_{31} \\ \phi_{13}\bar{\phi}_{32} \end{bmatrix}, \quad Z_4 = \begin{bmatrix} \phi_{21}\bar{\phi}_{33} \\ \phi_{22}\bar{\phi}_{33} \\ \phi_{23}\bar{\phi}_{31} \\ \phi_{23}\bar{\phi}_{32} \end{bmatrix}. \qquad (125)$$

Variations of the scattering matrix δS are still given by Eq. (83), with δQ modified in view of the form of (123). Then defining vectors

$$\varphi_1 = (\varphi_{1j}) \equiv \bar{s}_{22}\psi_1 - \bar{s}_{21}\psi_2, \quad \varphi_2 = (\varphi_{2j}) \equiv -\bar{s}_{12}\psi_1 + \bar{s}_{11}\psi_2, \qquad (126)$$

$$\bar{\varphi}_1 = (\bar{\varphi}_{1j}) \equiv s_{22}\bar{\psi}_1 - s_{12}\bar{\psi}_2, \quad \bar{\varphi}_2 = (\bar{\varphi}_{2j}) \equiv -s_{21}\bar{\psi}_1 + s_{11}\bar{\psi}_2, \qquad (127)$$

we find that variations of the scattering data are

$$\delta\rho_1(\xi) = \frac{1}{s_{33}^2(\xi)}\left\langle \Omega_1(\xi;x), \begin{bmatrix} \delta u(x) \\ \delta v(x) \\ \delta u^*(x) \\ \delta v^*(x) \end{bmatrix} \right\rangle, \quad \delta\rho_2(\xi) = \frac{1}{s_{33}^2(\xi)}\left\langle \Omega_2(\xi;x), \begin{bmatrix} \delta u(x) \\ \delta v(x) \\ \delta u^*(x) \\ \delta v^*(x) \end{bmatrix} \right\rangle, \qquad (128)$$

$$\delta\bar{\rho}_1(\xi) = \frac{1}{\bar{s}_{33}^2(\xi)}\left\langle \Omega_3(\xi;x), \begin{bmatrix} \delta u(x) \\ \delta v(x) \\ \delta u^*(x) \\ \delta v^*(x) \end{bmatrix} \right\rangle, \quad \delta\bar{\rho}_2(\xi) = \frac{1}{\bar{s}_{33}^2(\xi)}\left\langle \Omega_4(\xi;x), \begin{bmatrix} \delta u(x) \\ \delta v(x) \\ \delta u^*(x) \\ \delta v^*(x) \end{bmatrix} \right\rangle, \qquad (129)$$

where

$$\Omega_1 = \begin{bmatrix} \bar{\psi}_{31}\varphi_{13} \\ \bar{\psi}_{32}\varphi_{13} \\ -\bar{\psi}_{33}\varphi_{11} \\ -\bar{\psi}_{33}\varphi_{12} \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} \bar{\psi}_{31}\varphi_{23} \\ \bar{\psi}_{32}\varphi_{23} \\ -\bar{\psi}_{33}\varphi_{21} \\ -\bar{\psi}_{33}\varphi_{22} \end{bmatrix}, \quad \Omega_3 = \begin{bmatrix} -\bar{\varphi}_{11}\psi_{33} \\ -\bar{\varphi}_{12}\psi_{33} \\ \bar{\varphi}_{13}\psi_{31} \\ \bar{\varphi}_{13}\psi_{32} \end{bmatrix}, \quad \Omega_4 = \begin{bmatrix} -\bar{\varphi}_{21}\psi_{33} \\ -\bar{\varphi}_{22}\psi_{33} \\ \bar{\varphi}_{23}\psi_{31} \\ \bar{\varphi}_{23}\psi_{32} \end{bmatrix}. \qquad (130)$$

For Manakov equations, the time evolution operator of the Lax pair is

$$Y_t = -2i\zeta^2\Lambda Y + V(\zeta, u)Y, \qquad (131)$$

where V goes to zero as x approaches infinity. Introducing "time-dependent" Jost functions and adjoint Jost functions such as

$$\Phi^{(t)} = \Phi e^{-2i\zeta^2\Lambda t}, \quad \bar{\Phi}^{(t)} = e^{2i\zeta^2\Lambda t}\bar{\Phi}, \qquad (132)$$

and use them to replace the original Jost functions and adjoint Jost functions in the Z_k and Ω_k definitions (125) and (130), the resulting functions $Z_{1,2,3,4}^{(t)}$ and $\Omega_{1,2,3,4}^{(t)}$ would be squared eigenfunctions and adjoint squared eigenfunctions of the Manakov equations. Equivalently, $\{Z_1 e^{-4i\zeta^2 t}, Z_2 e^{-4i\zeta^2 t}, Z_3 e^{4i\zeta^2 t}, Z_4 e^{4i\zeta^2 t}\}$ are squared eigenfunctions, and $\{\Omega_1 e^{4i\zeta^2 t}, \Omega_2 e^{4i\zeta^2 t}, \Omega_3 e^{-4i\zeta^2 t}, \Omega_4 e^{-4i\zeta^2 t}\}$ adjoint squared eigenfunctions, of the Manakov equations.

It is noted that squared eigenfunctions for the Manakov equations have been given in [18] under different notations and derivations. Our expressions above are more explicit and easier to use.

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[1] D. J. Kaup, A perturbation expansion for the Zakharov-Shabat inverse scattering transform, SIAM J. Appl. Math. 31, 121 (1976).

[2] D. J. Kaup, Closure of the squared Zakharov-Shabat eigenstates, J. Math. Anal. Appl. 54, 849864 (1976).

[3] A. C. Newell, The inverse scattering transform, in *Solitons*, Topics in Current Physics, Vol. 17, edited by R. K. Bullough and P. J. Caudrey, Springer-Verlag, New York pp. 177242 (1980).

[4] D.J. Kaup and T.I. Lakoba, "Squared eigenfunctions of the Massive Thirring Model in laboratory coordinates", J. Math. Phys. 37 308 (1996).

[5] X. Chen and J. Yang, "A direct perturbation theory for solitons of the derivative nonlinear Schroedinger equation and the modified nonlinear Schroedinger equation." Phys. Rev. E. 65, 066608 (2002).

[6] D.J. Kaup, T.I. Lakoba and Y. Matsuno, "Perturbation theory for the Benjamin-Ono equation", Inverse Problems 15 215-240 (1999).

[7] R. L. Herman, A direct approach to studying soliton perturbations, J. Phys. A 23, 23272362 (1990).

[8] M.J. Ablowitz, D.J. Kaup, A.C. Newell, and H. Segur, The inverse scattering transformFourier analysis for nonlinear problems, Stud. Appl. Math. 53, 249336 (1974).

[9] A. S. Fokas and P. M. Santini, The recursion operator of the Kadomtsev-Petviashvili equation and the squared eigenfunctions of the Schrdinger operator, Stud. Appl. Math. 75, pp. 179185 (1986).

[10] J. Yang, "Complete eigenfunctions of linearized integrable equations expanded around a soliton solution". J. Math. Phys. 41, 6614 (2000).

[11] J. Yang, "Eigenfunctions of linearized integrable equations expanded around an arbitrary solution." Stud. Appl. Math. 108, 145 (2002).

[12] D.J. Kaup, Perturbation theory for solitons in optical fibers, Phys. Rev. A 42, 5689 (1990).

[13] Y. Zeng, W. Ma and R. Lin, "Integration of the soliton hierarchy with self-consistent sources", J. Math. Phys. 41, 5453-5489 (2000).

[14] H. Wang, X. Hu and Gegenhasi, "2D Toda lattice equation with self-consistent sources: Casoratian type solutions, bilinear Bcklund transformation and Lax pair", J. Computational and Applied Math. 202, 133-143 (2007).

[15] J. Yang, "Stable embedded solitons." Phys. Rev. Lett. 91, 143903 (2003).

[16] J. Yang and X. Chen, "Linearization operators of solitons in the derivative nonlinear Schroedinger hierarchy", *Trends in Soliton Research*, pp 15-27, Editor L.V. Chen, Nova Publishers, 2006.

[17] D.J. Kaup, "The Squared Eigenstates of the Sine-Gordon Eigenvalue Problem", J. Math. Phys. 25, 2467-71 (1984).

[18] V.S. Gerdjikov, "Basic aspects of soliton theory", in *Geometry, Integrability and Quantization - VI*, Eds. I. M. Mladenov, A. C. Hirshfeld, Softex, Sofia, pp. 78-122 (2005).

[19] A.S. Fokas, "A symmetry approach to exactly solvable evolution equations", J. Math. Phys. 21, 1318-1325 (1980).

[20] A. Sergyeyev and D. Demskoi, "Sasa-Satsuma (complex modified Kortewegde Vries II) and the complex sine-Gordon II equation revisited: Recursion operators, nonlocal symmetries, and more", J. Math. Phys. 48, 042702 (2007).

[21] A. Hasegawa and Y. Kodama, *Solitons in Optical Communications*, Clarendon, Oxford (1995).

[22] G.P. Agrawal, *Nonlinear Fiber Optics*, Academic Press, San Diego (1989).

[23] N. Sasa and J. Satsuma, "New type of soliton solutions for a higher-order nonlinear Schrödinger equation", J. Phys. Soc. Japan, 60, 409-417 (1991).

[24] J. Yang, B.A. Malomed, and D.J. Kaup, "Embedded solitons in second-harmonic-generating systems." Phys. Rev. Lett. 83, 1958 (1999).

[25] V.S. Shchesnovich and J. Yang, "General soliton matrices in the Riemann-Hilbert problem for integrable nonlinear equations." J. Math. Phys. 44, 4604 (2003).

[26] S.V. Manakov, "On the Theory of Two-dimensional Stationary Self-focusing of Electromagnetic Waves", Zh. Eksp. Teor. Fiz 65, 1392 (1973) [Sov. Phys. JETP 38, 248-253 (1974)].